Trigonometrijski izrazi

21. listopada 2017.

Uvod/teorijske osnove

Trigonometrijski izrazi česta su tema na natjecanjima za 3. razred srednje škole u obje kategorije. Iako je za rješavanje zadataka uglavnom potrebno samo znanje naučeno na nastavi, iz ovog područja postoje mnogi zanimljivi zadaci. Podsjetimo se bitnih formula:

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}$$
$$\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos(x)}{2}$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$
$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

Zadatci i rješenja

Zadatak 1.
Pojednostavi:

$$\sqrt{\sin^4(x) + 4\cos^2(x)} - \sqrt{\cos^4(x) + 4\sin^2(x)}$$

Rješenje.
Iz zadane jednadžbe dobivamo:

$$\begin{aligned}\sqrt{\sin^4(x) + 4\cos^2(x)} - \sqrt{\cos^4(x) + 4\sin^2(x)} = \\ = \sqrt{\sin^4(x) + 4(1 - sin^2(x))} - \sqrt{\cos^4(x) + 4(1 - \cos^2(x))} \\ = \sqrt{(2 - \sin^2(x))^2} - \sqrt{(2 - \cos^2(x))^2} \\ = \cos^2(x) - \sin^2(x) \\ = \cos(2x)\end{aligned}$$

Zadatak 2.
Odredite sumu rješenja jednadžbe

$$\sin(3x) + \cos(2x) + 2\cos^2(x) = 0$$

u segmentu $[-\pi, 2\pi]$

Rješenje.
Prvo:

$$\sin(3x) = \sin(2x + x) = ... = 3\sin(x) - 4\sin^3(x)$$

Također:

$$\cos(2x) = 1 - \sin^2(x)$$

Sada je cijela jednadžba ekvivalentna s:

$$3\sin(x) - 4\sin^3(x) + 1 - \sin^2(x) - 2\sin^2(x) = 0$$

Odnosno:

$$(1 + \sin(x))(3 - 4\sin^2(x)) = 0)$$

Rješenja su:

1. $\sin(x) = -1, x = -\frac{\pi}{2} + 2k\pi$
2. $\sin(x) = \frac{\sqrt{3}}{2}, x = \frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi$
3. $\sin(x) = -\frac{\sqrt{3}}{2}, x = -\frac{\pi}{3} + 2k\pi, -\frac{2\pi}{3} + 2k\pi$

U danom segmentu su rješenja $x_1 = -\frac{\pi}{2}$, $x_2 = \frac{3\pi}{2}$, $x_3 = \frac{\pi}{3}$, $x_4 = \frac{2\pi}{3}$, $x_5 = -\frac{\pi}{3}$, $x_6 = \frac{5\pi}{3}$, $x_7 = -\frac{2\pi}{3}$, $x_8 = \frac{4\pi}{3}$. Tražena suma je π.

Zadatak 3.
Dokažite da vrijedi:

$$1 - \text{ctg}(23^\circ) = \frac{2}{1 - \text{ctg}(22^\circ)}$$

Rješenje.
Pokazat ćemo da vrijedi:

$$(1 - \text{ctg}(23^\circ))(1 - \text{ctg}(22^\circ)) = 2$$

$$\begin{aligned}(1 - \text{ctg}(23^\circ))(1 - \text{ctg}(22^\circ)) &= (1 - \frac{\cos(23^\circ)}{\sin(23^\circ)})(1 - \frac{\cos(22^\circ)}{\sin(22^\circ)}) \\ &= \frac{\sin(23^\circ) - \cos(23^\circ)}{\sin(23^\circ)} \frac{\sin(22^\circ) - \cos(22^\circ)}{\sin(22^\circ)} \\ &= \frac{\sqrt{2}\sin(23^\circ - 45^\circ)\sqrt{2}\sin(22^\circ - 45^\circ)}{\sin(23^\circ) * \sin(22^\circ)} \\ &= \frac{2\sin(-22^\circ)\sin(-23^\circ)}{\sin(23^\circ) * \sin(22^\circ)} \\ &= 2\end{aligned}$$

Zadatak 4.
Izračunaj:

$$\sin(20°)\sin(40°)\sin(80°)$$

Rješenje.
Koristimo formulu pretvorbe umnoška u zbroj dva puta:

$$\begin{aligned}
\sin(20^\circ)\sin(40^\circ)\sin(80^\circ) &= \frac{1}{2}(\cos(20^\circ) - \cos(60^\circ))\sin(80^\circ) \\
&= \frac{1}{2}(\cos(20^\circ) - \frac{1}{2})\sin(80^\circ) \\
&= \frac{1}{2}\cos(20^\circ)\sin(80^\circ) - \frac{1}{4}\sin(80^\circ) \\
&= \frac{1}{4}(\sin(100^\circ) + \sin(60^\circ)) - \frac{1}{4}\sin(80^\circ) \\
&= \frac{1}{4}(\sin(80^\circ) + \frac{\sqrt{3}}{2}) - \frac{1}{4}\sin(80^\circ) \\
&= \frac{\sqrt{3}}{8}
\end{aligned}$$

Zadatak 5.
Pokažite da u trokutu vrijedi:

$$\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 4\sin(\alpha)\sin(\beta)\sin(\gamma)$$

Rješenje.
Koristeći formule pretvorbe zbroja u umnožak i činjenicu da $\alpha + \beta + \gamma = 180^\circ$ rješavamo:

$$\begin{aligned}
\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) &= 2\sin(\alpha + \beta)\cos(\alpha - \beta) + \sin(2\gamma) \\
&= 2\sin(\gamma)\cos(\alpha - \beta) + 2\sin(\gamma)\cos(\gamma) \\
&= 2\sin(\gamma)[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\
&= 2\sin(\gamma)(-2\sin(\alpha)\sin(-\beta) \\
&= 4\sin(\alpha)\sin(\beta)\sin(\gamma))
\end{aligned}$$

Zadatak 6.
Izračunaj umnožak:

$$(1 + \tan(1°))(1 + \tan(2°))...(1 + \tan(45°))$$

Rješenje.
Ovaj zadatak zahtjeva dozu dosjetljivosti, česta ideja u ovakvim situacijama je neka vrsta uparivanja. Stoga promatrajmo umnožak

$$(1 + \tan(\alpha))(1 + \tan(45^\circ - \alpha)) = 1 + \tan(\alpha) + \tan(45^\circ - \alpha) + \tan(\alpha)\tan(45^\circ - \alpha)$$

Desni dio jednakosti sadrži dijelove koji nas podsjećaju na tangesns zbroja:

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

Iz te formule slijedi:

$$\tan(\alpha) + \tan(45^\circ - \alpha) = \tan(45^\circ)(1 - \tan(\alpha)\tan(45^\circ - \alpha))$$

Odnosno:

$$\tan(\alpha) + \tan(45^\circ - \alpha) = 1 - \tan(\alpha)\tan(45^\circ - \alpha)$$

Kada ovu činjenicu ubacimo u prvu jednakost dobivamo:

$$(1 + \tan(\alpha))(1 + \tan(45^\circ - \alpha)) = 2$$

Grupirajući umnoške u 22 para za $\alpha = 1^\circ, 2^\circ, ..., 22^\circ$ i shvativši da zadnja zagrada u prvotnome umnošku ima vrijednost 2 dobivamo da je ukupan umnožak jednak 2^{23}.

Zadatak 7.
Izračunaj:

$$\tan(9°) + \tan(81°) + \tan(117°) + \tan(153°)$$

Rješenje.
Koristimo svođenje na prvi kvadrant, funkcije zbroja i dvostrukog kuta. Podsjetimo se svođenja na prvi kvadrant potrebnog za ovaj zadatak:

$$\tan(180^\circ - x) = -\tan(x)$$

$$\cos(90^\circ - x) = \sin(x)$$

$$\begin{aligned}
\tan(9^\circ) + \tan(81^\circ) + \tan(117^\circ) + \tan(153^\circ) &= \tan(9^\circ) + \tan(81^\circ) - \tan(63^\circ) - \tan(27^\circ) \\
&= \frac{\sin(9^\circ)}{\cos(9^\circ)} + \frac{\sin(81^\circ)}{\cos(81^\circ)} - \left[\frac{\sin(63^\circ)}{\cos(63^\circ)} + \frac{\sin(27^\circ)}{\cos(27^\circ)}\right] \\
&= \frac{\sin(9^\circ)\cos(81^\circ) + \sin(81^\circ)\cos(9^\circ)}{\cos(9^\circ)\cos(81^\circ)} - \left[\frac{\sin(63^\circ)\cos(27^\circ) + \sin(27^\circ)\cos(63^\circ)}{\cos(63^\circ)\cos(27^\circ)}\right] \\
&= \frac{\sin(9^\circ + 81^\circ)}{\cos(9^\circ)\cos(81^\circ)} - \frac{\sin(63^\circ + 27^\circ)}{\cos(27^\circ)\cos(63^\circ)} \\
&= \frac{\sin(90^\circ)}{\cos(9^\circ)\cos(90^\circ - 9^\circ)} - \frac{\sin(90^\circ)}{\cos(27^\circ)\cos(90^\circ - 27^\circ)} \\
&= \frac{1}{\cos(9^\circ)\sin(9^\circ)} - \frac{1}{\cos(27^\circ)\sin(27^\circ)} \\
&= 2\left[\frac{1}{\sin(18^\circ)} - \frac{1}{\sin(54^\circ)}\right] = 2\left[\frac{\sin(54^\circ) - \sin(18^\circ)}{\sin(18^\circ)\sin(54^\circ)}\right] \\
&= 2\left[\frac{2\cos(36^\circ)\sin(18^\circ)}{\sin(18^\circ)\sin(54^\circ)}\right] = 4\frac{\cos(36^\circ)}{\sin(54^\circ)} \\
&= 4\frac{\cos(90^\circ - 54^\circ)}{\sin(54^\circ)} = 4
\end{aligned}$$

Zadatak 8.
Pojednostavi:

$$\frac{\sin(5x)\cos(x) - \cos(3x)\sin(x)}{\cos^2(3x) - \cos^2(x)}$$

Rješenje.

$$\begin{aligned}
\frac{\sin(5x)\cos(x) - \cos(3x)\sin(x)}{\cos^2(3x) - \cos^2(x)} &= \frac{\frac{1}{2}[\sin(6x) + \sin(4x)] - \frac{1}{2}[\sin(4x) - \sin(2x)]}{[\cos(3x) - \cos(x)][\cos(3x) + \cos(x)]} \\
&= \frac{\sin(6x) + \sin(2x)}{2[-2\sin(2x)\sin(x)][2\cos(2x)\cos(x)]} \\
&= -\frac{\sin(6x) + \sin(2x)}{2\sin(4x)\sin(2x)} \\
&= -\frac{2\sin(4x)\cos(2x)}{2\sin(4x)\sin(2x)} \\
&= -\cot(2x)
\end{aligned}$$

Zadatak 9.
Ako je $\sin(\alpha) + \sin(\beta) = 2\sin(\alpha + \beta)$, $\alpha + \beta \neq 2k\pi (k \in Z)$, odredi $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)$.

Rješenje.

$$\sin(\alpha) + \sin(\beta) = 2\sin(\alpha + \beta)$$

Na lijevoj strani koristimo formulu pretvorbe zbroja u umnožak, a desnu raspisujemo kao sin dvostrukog kuta:

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 2 * 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\left(\frac{\alpha-\beta}{2}\right) = 2\cos\left(\frac{\alpha+\beta}{2}\right)$$

Dalje obje strane raspisujemo kao funkciju razlike, odnosno zbroja:

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) = 2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) = 3\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$\frac{\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)} = \frac{1}{3}$$

Dakle,

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{1}{3}$$

Zadatak 10.
Dokaži da vrijedi:

$$\sin(x+y)\sin(x-y) = \sin^2(x) - \sin^2(y)$$

Rješenje.
Raspisujemo lijevu stranu navedene jednakosti:

$$\begin{aligned}
\sin(x+y)\sin(x-y) &= [\sin(x)\cos(y) + \cos(x)\sin(y))(\sin(x)\cos(y) - \cos(x)\sin(y)] \\
&= \sin^2(x)\cos^2(y) - \cos^2(x)\sin^2(y) \\
&= \sin^2(x)[1-\sin^2(y)] - [1-\sin^2(x)]\sin^2(y) \\
&= \sin^2(x) - \sin^2(x)\sin^2(y) - \sin^2(y) + \sin^2(x)\sin^2(y) \\
&= \sin^2(x) - \sin^2(y)
\end{aligned}$$

Zadatak 11.
Izračunaj $\cos(4x)$, ako vrijedi:

$$\frac{1}{\tan^2 x} + \frac{1}{\cot^2 x} + \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = 7$$

Rješenje.
Dani izraz raspisujemo i svodimo na zajednički nazivnik:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\sin^2 x} + \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = 7$$

$$\frac{\cos^2 x + 1}{\sin^2 x} + \frac{\sin^2 x + 1}{\cos^2 x} = 7$$

$$\frac{1-\sin^2 x + 1}{\sin^2 x} + \frac{1-\cos^2 x + 1}{\cos^2 x} = 7$$

$$-1 + \frac{2}{\sin^2 x} - 1 + \frac{2}{\cos^2 x} = 7$$

Množenjem cijelog izraza zajedničkim nazivnikom dobivamo:

$$2(\cos^2 x + \sin^2 x) = 9\sin^2 x\cos^2 x$$

$$\sin^2 x\cos^2 x = \frac{2}{9}$$

Izraz zatim množimo sa 4 da bismo dobili sin(2x) što potom raspisujemo kao sin polovičnog kuta:

$$[2\sin(x)\cos(x)]^2 = \frac{8}{9}$$

$$\sin^2\frac{4x}{2} = \frac{8}{9}$$

$$\frac{1-\cos(4x)}{2} = \frac{8}{9}$$

$$\cos(4x) = \frac{-7}{9}$$

Zadatak 12.
Dokaži sljedeću jednakost:

$$\cos^3(x)\sin^2(x) = \frac{1}{16}[2\cos(x) - \cos(3x) - \cos(5x)]$$

Rješenje.

$$
\begin{aligned}
\cos^3(x)\sin^2(x) &= [\sin(x)\cos(x)]^2\cos(x)\\
&= \left[\frac{1}{2}\sin(2x)]^2\cos(x)\right]\\
&= \frac{1}{4}\sin(2x)[\sin(2x)\cos(x)]\\
&= \frac{1}{4}\sin(2x)\frac{1}{2}[\sin(3x)+\sin(x)]\\
&= \frac{1}{8}[\sin(2x)\sin(3x)+\sin(2x)\sin(x)]\\
&= \frac{1}{8}\left\{-\frac{1}{2}(\cos(5x)-\cos(x)+\left[-\frac{1}{2}(\cos(3x)+\cos(x)\right]\right\}\\
&= \frac{1}{16}[2\cos(x)-\cos(3x)-\cos(5x)]
\end{aligned}
$$