JS12	zajednička grupa

Predavanja subotom
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Mix za sve uzraste

Hrvoje Radoš

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Mladi nadareni matematičari "Marin Getaldić"

U ovom životu nije teško mrijeti. Izgraditi život, daleko je teže. - Vladimir Majakovski

Zagrijavanje

1. Faktoriziraj $f(x) = x^5 - 2x^4 + x^3 + x^2 - 2x + 1$.

2. Ako je $x + y + z = 6$, $x, y, z \geq 0$, dokažite da je onda $x^2 + y^2 + z^2 \geq 12$.

Juniori

3. Odredite sva realna rješenja sustava jednadžbi:

$$2a^2 - 2ab + b^2 = a$$
$$4a^2 - 5ab + 2b^2 = b$$

4. Neka su x, y, z različiti realni brojevi. Dokažite:

$$\sqrt[3]{x-y} + \sqrt[3]{y-z} + \sqrt[3]{z-x} \neq 0$$

5. Riješite sustav:

$$(b+c+d)^{2010} = a$$
$$(c+d+a)^{2010} = b$$
$$(d+a+b)^{2010} = c$$
$$(a+b+c)^{2010} = d$$

$a, b, c, d \in \mathbb{R}$

6. Neka je $n \in \mathbb{N}$. Dokažite da vrijedi:

(a) $\left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right)^n$. (b) $\left(1 + \frac{1}{n}\right)^n < 4$.

7. Neka su a, b i c pozitivni realni brojevi. Dokaži da vrijedi

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} \geq \frac{3a+2b-c}{4}.$$

8. Odredite minimalnu vrijednost od

$$\frac{a+3c}{a+2b+c} + \frac{4b}{a+b+2c} - \frac{8c}{a+b+3c}$$

$a, b, c \in \mathbb{R}^+$

Seniori

9. Dan je niz za koji vrijedi:

$$(3 - a_{n+1})(6 + a_n) = 18\,, a_0 = 3$$

Odredite:

$$\sum_{i=0}^{n} \frac{1}{a_i}$$

10. Neka $f : \mathbb{R} \to \mathbb{R}$. $f_{(1)} = 1$ i $\forall x \in \mathbb{R}$

$$f_{(x+5)} \geq f_{(x)} + 5$$

$$f_{(x+1)} \leq f_{(x)} + 1$$

Ako $g_{(x)} = f_{(x)} + 1 - x$ odredite $g_{(2002)}$

11. Neka $\frac{3}{2} \leq x \leq 5$. Dokažite:

$$2\sqrt{x+1} + \sqrt{2x-3} + \sqrt{15-3x} < 2\sqrt{19}$$

12. Odredite sve polinome $p : \mathbb{R} \to \mathbb{R}$ koji zadovoljavaju jednadžbu

$$(x+1)f_{(x-1)} = (x-2)f_{(x)}$$

13. Odredite sve $f : \mathbb{R}^+ \to \mathbb{R}^+$ takve da:

$$(f \circ f)_{(x)} = 6x - f_{(x)}$$

1. Rješenja

Zagrijavanje

1. Zadatak se može riješiti s pogađanjem nultočaka polinoma. Najbolje je provjeriti jesu li djelitelji slobodnog koeficijenta nultočke (ovo nije teorem, ali dosta dobro pogađanje). S obzirom da je slobodni koeficijent 1 ima smisla pogoditi 1 i -1 kao nultočke. I uvršavanjem ispadne da su 1 i -1 uistinu nultočke, stoga dijeljenjem polinom dobijemo:

$$f_{(x)} = (x+1)(x-1)(x^3 - 2x^2 + 2x - 1)$$

Istim način pogodimo da polinom $x^3 - 2x^2 + 2x - 1$ ima 1 kao nultočku, stoga imamo

$$f_{(x)} = (x+1)(x-1)^2(x^2 - x + 1)$$

I na kraju znamo da je $x^2 - x + 1$ nemoguće dalje faktorizirati jer je diskriminanta te kvadratne jednadžbe negativna.

2. KAGH

$$\sqrt{\frac{x^2+y^2+z^2}{3}} \geq \frac{x+y+z}{3} = 2$$
$$\frac{x^2+y^2+z^2}{3} \geq 4$$
$$x^2+y^2+z^2 \geq 12$$

Juniori

3. Županijsko 2. razred A varijanta 1.zadatak

4. $X = (\sqrt[3]{x-y} + \sqrt[3]{y-z} + \sqrt[3]{z-y})$
Vrijedi faktorizacija $a^3 - b^3 - c^3 = (a+b+c) \cdot Y$ gdje je Y simetrični algebarski izraz u varijablama a, b, i c.
Ako uvrstimo $a = \sqrt[3]{x-y},\ b = \sqrt[3]{y-z},\ c = \sqrt[3]{z-y}$ Vrijedi $x - y + y - z + z - y - 3abc = (\sqrt[3]{x-y} + \sqrt[3]{y-z} + \sqrt[3]{z-y}) \cdot Y \iff -3abc = X \cdot Y$
Pretpostavimo da vrijednost izraza X jest jednaka 0. Budući da su x, y i z međusobno različiti, a b i c su različiti od 0, pa njihov umnožak ne može biti nula, a budući da je $X = 0$, -3abc mora biti jednako 0. Dobili smo kontrakdikciju pa znamo da pretpostavka ne vrijedi tj. $X \neq 0$, što je i trebalo pokazati.

5. link

6. (a)

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n \cdot 1} \overset{\text{A-G}}{\leq} \frac{n(1+\frac{1}{n})+1}{n+1} = \frac{n+2}{n+1} = 1 + \frac{1}{n+1}$$

(b)

$$1 + \frac{1}{n} = \frac{n+1}{n} = \frac{n+1}{1+1+\cdots+1+\frac{1}{2}+\frac{1}{2}} \overset{\text{A-H}}{\leq} \sqrt[n+1]{1 \cdot 1 \cdots 1 \cdot 2 \cdot 2} = \sqrt[n+1]{4} \leq \sqrt[n]{4}.$$

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8. rješenje:

$a+2b+c \quad a+b+2c \quad a+b+3c$

Solution The answer is $12\sqrt{2}-17$. Set

$$\begin{cases} x = a+2b+c, \\ y = a+b+2c, \\ z = a+b+3c. \end{cases}$$

It is easy to see that $z-y=c$ and $x-y=b-c$, giving $x-y=b-(z-y)$, or $b=x+z-2y$. We note that $a+3c=2y-x$. By the AM-GM Inequality, it follows that

$$\begin{aligned} &\frac{a+3c}{a+2b+c}+\frac{4b}{a+b+2c}-\frac{8c}{a+b+3c} \\ &= \frac{2y-x}{x}+\frac{4(x+z-2y)}{y}-\frac{8(z-y)}{z} \\ &= -17+2\frac{y}{x}+4\frac{x}{y}+4\frac{z}{y}+8\frac{y}{z} \\ &\geqslant -17+2\sqrt{8}+2\sqrt{32} = -17+12\sqrt{2}. \end{aligned}$$

The equality holds if and only if $\frac{2y}{x}=\frac{4x}{y}$ and $\frac{4z}{y}=\frac{8y}{z}$, or $4x^2=2y^2=z^2$. Hence the equality holds if and only if

$$\begin{cases} a+b+2c=\sqrt{2}(a+2b+c), \\ a+b+3c=2(a+2b+c). \end{cases}$$

Solving the above system of equations for b and c in terms of a gives

$$\begin{cases} b=(1+\sqrt{2})a, \\ c=(4+3\sqrt{2})a. \end{cases}$$

We conclude that

$$\frac{a+3c}{a+2b+c}+\frac{4b}{a+b+2c}-\frac{8c}{a+b+3c}$$

has minimum value $12\sqrt{2}-17$ if and only if $(a, b, c)=(a, (1+\sqrt{2})a, (4+3\sqrt{2})a)$.

Seniori

9. rješenje:

Solution Set $b_n = \frac{1}{a_n}$, $n = 0, 1, 2, \cdots$, then $\left(3 - \frac{1}{b_{n+1}}\right)\left(6 + \frac{1}{b_n}\right) = 18$, namely,

$$3b_{n+1} - 6b_n - 1 = 0.$$

Hence $b_{n+1} = 2b_n + \frac{1}{3}$, or $b_{n+1} + \frac{1}{3} = 2\left(b_n + \frac{1}{3}\right)$. So $\left\{b_n + \frac{1}{3}\right\}$ is a geometric progression with common ratio 2. Thus

$$b_n + \frac{1}{3} = 2^n\left(b_0 + \frac{1}{3}\right) = 2^n\left(\frac{1}{a_0} + \frac{1}{3}\right) = \frac{1}{3} \times 2^{n+1},$$

$$b_n = \frac{1}{3}(2^{n+1} - 1).$$

Therefore,

$$\sum_{i=0}^{n} \frac{1}{a_i} = \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} \frac{1}{3}(2^{i+1} - 1)$$

$$= \frac{1}{3}\left[\frac{2(2^{n+1} - 1)}{2 - 1} - (n+1)\right]$$

$$= \frac{1}{3}(2^{n+2} - n - 3).$$

10. rješenje:

Solution We determine $f(2\,002)$ first. From the conditions given, we have

$$\begin{aligned} f(x)+5 &\leqslant f(x+5) \leqslant f(x+4)+1 \\ &\leqslant f(x+3)+2 \leqslant f(x+2)+3 \\ &\leqslant f(x+1)+4 \leqslant f(x)+5. \end{aligned}$$

Thus the equality holds for all. So we have $f(x+1)=f(x)+1$.

Hence, from $f(1)=1$, we get $f(2)=2$, $f(3)=3$, $\cdots$, $f(2\,002)=2\,002$. Therefore, $g(2\,002)=f(2\,002)+1-2\,002=1$.

11. rješenje:

Solution By Cauchy's inequality, we have

$$\begin{aligned} & 2\sqrt{x+1}+\sqrt{2x-3}+\sqrt{15-3x} \\ = & \sqrt{x+1}+\sqrt{x+1}+\sqrt{2x-3}+\sqrt{15-3x} \\ \leqslant & \sqrt{[(x+1)+(x+1)+(2x-3)+(15-3x)](1^2+1^2+1^2+1^2)} \\ = & 2\sqrt{x+14} \leqslant 2\sqrt{19}, \end{aligned}$$

and the equality holds if and only $\sqrt{x+1}=\sqrt{2x-3}=\sqrt{15-3x}$ and $x=5$. But this is impossible. So $2\sqrt{x+1}+\sqrt{2x-3}+\sqrt{15-3x}<2\sqrt{19}$.

Remark Some student contestants used the estimate that

$$\begin{aligned} & 2\sqrt{x+1}+\sqrt{2x-3}+\sqrt{15-3x} \\ \leqslant & \sqrt{[(x+1)+(2x-3)+(15-3x)](2^2+1^2+1^2)} \\ = & \sqrt{78}, \end{aligned}$$

but this does not give the value as required.

12. Uvrstimo neke vrijednosti:

- $x=0$

$$\begin{aligned} (0+1)\,f(-1) &= (0-2)\,f(0) \\ f(-1) &= -2f(0) \end{aligned}$$

- $x=1$

$$2f(0)=-f(1)$$

- $x = -1$

$$0 = -3f(-1)$$
$$f(-1) = 0$$

- $x = 2$

$$3f(1) = 0$$
$$f(1) = 0$$

Dakle, dobili smo da je $f(1) = f(-1) = f(0) = 0$, pa je polinom f oblika

$$f(x) = x(x-1)(x+1) \cdot g(x),$$

za neki polinom g. Uvrstimo to u početnu jednakost.

$$(x+1) f(x-1) = (x-2) f(x)$$
$$(x+1)(x-1)(x-2) x g(x-1) = (x-2) x (x-1)(x+1) g(x)$$
$$g(x-1) = g(x),$$

za $x \neq -1, 0, 1, 2$. Uvrštavanjem imamo $g(2) = g(3) = g(4) = g(5) = \ldots$. Dokažimo induktivno da je $g(n) = g(2)$, za svaki $n \geqslant 2$.

Baza indukcije.
Za $n = 2$ imamo
$$g(2) = g(2).$$
Dakle, baza indukcije vrijedi.

Pretpostavka indukcije.
Pretpostavimo da za neki $n \geqslant 2$ vrijedi $g(n) = g(2)$. *Korak indukcije.*

$$g(n+1) = g(n+1-1) = g(n) = g(2).$$

Definirajmo sada $h(x) = g(x) - g(2)$. Polinom $g(x)$ i konstantni polinom $g(2)$ su jednaki, njihova razlika ima beskonačno mnogo nultočaka, pa se radi o nulpolinomu.

13. link